Lecture Notes on Constant Elasticity Functions

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1 CES Utility

In many economic textbooks the constant-elasticity-of-substitution (CES) utility function is defined as:

$$U(x,y) = (\alpha x^\rho + (1-\alpha)y^\rho)^{1/\rho}$$

It is a tedious but straight-forward application of Lagrangian calculus to demonstrate that the associated demand functions are:

$$x(p_x,p_y,M) = \left(\frac{\alpha}{p_x}\right)^\sigma \frac{M}{\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma}}$$

and

$$y(p_x,p_y,M) = \left(\frac{1-\alpha}{p_y}\right)^\sigma \frac{M}{\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma}}.$$

The corresponding indirect utility function has is:

$$V(p_x,p_y,M) = M\left(\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma}\right)^{\frac{1}{\sigma-1}}$$

Note that $U(x,y)$ is linearly homogeneous:

$$U(\lambda x, \lambda y) = \lambda U(x,y)$$

This is a convenient cardinalization of utility, because percentage changes in U are equivalent to percentage Hicksian equivalent variations in income. Because U is linearly homogeneous, V is homogeneous of degree one in M:

$$V(p_x,p_y,\lambda M) = \lambda V(p_x,p_y,M)$$

and V is homogeneous of degree -1 in p:

$$V(\lambda p_x, \lambda p_y, M) = \frac{V(p_x,p_y,M)}{\lambda}.$$

Furthermore, linear homogeneity permits us to form an exact price index corresponding to the cost of a unit of utility:

$$e(p_x,p_y) = \left(\alpha^\sigma p_x^{1-\sigma} + (1-\alpha)^\sigma p_y^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

The indirect utility function can then be written:

$$V(p_x,p_y,M) = \frac{M}{e(p_x,p_y)}$$

Conceptually, this equation states that the utility which can be realized with income M and prices p_x and p_y is equal to the income level divided by the unit cost of utility. The key idea is that when the underlying is linearly homogeneous, utility can be represented like any other good in the economy. Put another way, without loss of generality, we can thing of each consumer demanding only one good.

2 CES Technology

In the representation of technology, we have a set of relationships which are directly analogous to the CES utility function. These relationships are based on the cost and compensated demand functions. If we have a CES production function of the form:

$$y(K, L) = \phi \left(\beta K^\rho + (1-\beta) L^\rho\right)^{1/\rho}$$

the unit cost function then has the form:

$$c(p_K, p_L) = \frac{1}{\phi} \left(\beta^\sigma p_K^{1-\sigma} + (1-\beta)^\sigma p_L^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

and associated demand functions are:

$$K(p_K, p_L, y) = \left(\frac{y}{\phi}\right) \left(\frac{\beta \phi c(p_K, p_L)}{p_K}\right)^\sigma$$

and

$$L(p_K, p_L, y) = \left(\frac{y}{\phi}\right) \left(\frac{(1-\beta) \phi c(p_K, p_L)}{p_L}\right)^\sigma.$$

In most large-scale applied general equilibrium models, we have many function parameters to specify with relatively few observations. The conventional approach is to *calibrate* functional parameters to a single benchmark equilibrium. For example, if we have benchmark estimates for output, labor, capital inputs and factor prices , we calibrate function coefficients by inverting the factor demand functions:[1]

$$\theta = \frac{\bar{p}_K \bar{K}}{\bar{p}_K \bar{K} + \bar{p}_L \bar{L}}, \quad \rho = \frac{\sigma - 1}{\sigma}, \quad \beta = \frac{\bar{p}_K \bar{K}^{1/\sigma}}{\bar{p}_K \bar{K}^{1/\sigma} + \bar{p}_L \bar{L}^{1/\sigma}}$$

and

$$\phi = \bar{y} \left[\beta \bar{K}^\rho + (1-\beta) \bar{L}^\rho\right]^{-1/\rho}$$

[1] I wish to thank Professor Olivier de La Grandville for correcting an error in the expression for β in an earlier version of these notes.

Exercises

1. Mikki once lived in Boulder and spent 30% of her income for rent, 10% for food and 60% for skiing. She then moved to Georgetown where rent and food prices are identical to Boulder. In Georgetown, however, Mikki discovered that the quality-adjusted cost of skiing was ten-times the cost of skiing in Boulder. She adopted a lifestyle in which she spend only 30% of her income on skiing. Suppose that her preferences are characterized by a CES utility function. What values of α and σ describe Mikki's utility function?

2. What fraction of Mikki's income does she spend on rent in Georgetown?

3. How much larger would Mikki's income need to be to compensate for the higher cost of skiing such that she would be indifferent between living in Boulder or Georgetown.

3 The Calibrated Share Form

Calibration formulae for CES functions are messy and difficult to remember. Consequently, the specification of function coefficients is complicated and error-prone. For applied work using calibrated functions, it is much easier to use the "calibrated share form" of the CES function. In the calibrated form, the cost and demand functions explicitly incorporate

- benchmark factor *demands*
- benchmark factor *prices*
- the elasticity of substitution
- benchmark *cost*
- benchmark *output*
- benchmark *value shares*

In this form, the production function is written:

$$y = \bar{y}\left[\theta\left(\frac{K}{\bar{K}}\right)^{\rho} + (1-\theta)\left(\frac{L}{\bar{L}}\right)^{\rho}\right]^{1/\rho}$$

The only *calibrated* parameter, θ, represents the value share of capital at the benchmark point, i.e.

$$\theta = \frac{\bar{p}_K\bar{K}}{\bar{p}_K\bar{K} + \bar{p}_L\bar{L}}$$

The corresponding cost functions in the calibrated form is written:

$$c(p_K, p_L) = \bar{c}\left[\theta\left(\frac{p_K}{\bar{p}_K}\right)^{1-\sigma} + (1-\theta)\left(\frac{p_L}{\bar{p}_L}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

where

$$\bar{c} = \bar{p}_L\bar{L} + \bar{p}_K\bar{K}$$

and the compensated demand functions are:

$$K(p_K, p_L, y) = \bar{K}\ \frac{y}{\bar{y}}\ \left(\frac{\bar{p}_K\ c}{p_K\ \bar{c}}\right)^{\sigma}$$

and

$$L(p_K, p_L, y) = \bar{L}\ \frac{y}{\bar{y}}\ \left(\frac{c\ \bar{p}_L}{\bar{c}\ p_L}\right)^{\sigma}$$

Normalizing the benchmark utility index to unity, the utility function in calibrated share form is written:

$$U(x, y) = \left[\theta\left(\frac{x}{\bar{x}}\right)^{\rho} + (1-\theta)\left(\frac{y}{\bar{y}}\right)^{\rho}\right]^{1/\rho}$$

The unit expenditure function can be written:

$$e(p_x, p_y) = \left[\theta\left(\frac{p_x}{\bar{p}_x}\right)^{1-\sigma} + (1-\theta)\left(\frac{p_y}{\bar{p}_y}\right)\right]^{\frac{1}{1-\sigma}},$$

the indirect utility function is:

$$V(p_x, p_y, M) = \frac{M}{\bar{M} e(p_x, p_y)},$$

and the demand functions are:

$$x(p_x, p_y, M) = \bar{x}\ V(p_x, p_y, M) \left(\frac{e(p_x, p_y)\bar{p}_x}{p_x} \right)^{\sigma}$$

and

$$y(p_x, p_y, M) = \bar{y}\ V(p_x, p_y, M) \left(\frac{e(p_x, p_y)\bar{p}_y}{p_y} \right)^{\sigma}.$$

The calibrated form extends directly to the n-factor case. An n-factor production function is written:

$$y = f(x) = \bar{y} \left[\sum_i \theta_i \left(\frac{x_i}{\bar{x}_i} \right)^{\rho} \right]^{1/\rho}$$

and has unit cost function:

$$C(p) = \bar{C} \left[\sum_i \theta_i \left(\frac{p_i}{\bar{p}_i} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

and compensated factor demands:

$$x_i = \bar{x}_i\ \frac{y}{\bar{y}}\ \left(\frac{C\ \bar{p}_i}{\bar{C}\ p_i} \right)^{\sigma}$$

Exercises

1. Show that given a generic CES utility function:

$$U(x, y) = (\alpha^{\rho} + (1 - \alpha)y^{\rho})^{1/\rho}$$

can be represented in share form using:

$$\bar{x} = 1,\ \bar{y} = 1,\ \bar{p}_x = t\alpha,\ \bar{p}_y = t(1 - \alpha),\ \bar{M} = t.$$

for any value of $t > 0$.

2. Consider the utility function defined:

$$U(x, y) = (x - a)^{\alpha}(y - b)^{1-\alpha}$$

A benchmark demand point with both prices equal and demand for y equal to twice the demand for x. Find values for which are consistent with optimal choice at the benchmark. Select these parameters so that the income elasticity of demand for x at the benchmark point equals 1.1.

3. Consider the utility function:

$$U(x, L) = (\alpha L^{\rho} + (1 - \alpha)x^{\rho})^{1/\rho}$$

which is maximized subject to the budget constraint:

$$p_x x = M + w(\bar{L} - L)$$

in which M is interpreted as non-wage income, w is the market wage rate. Assume a benchmark equilibrium in which prices for x and L are equal, demands for x and L are equal, and non-wage income equals one-half of expenditure on x. Find values of α and ρ consistent with these choices and for which the price elasticity of labor supply equals 0.2.

4. Consider a consumer with CES preferences over two goods. A price change makes the benchmark consumption bundle unaffordable, yet the consumer is indifferent. Graph the choice. Find an equation which determines the elasticity of substitution as a function of the benchmark value shares. (You can write down the equation, but it cannot be solved in closed form.)

5. Consider a model with three commodities, x, y and z. Preferences are CES. Benchmark demands and prices are equal for all goods. Find demands for x, y and z for a doubling in the price of x as a function of the elasticity of substitution.

6. Consider the same model in the immediately preceeding question, except assume that preferences are instead given by:

$$U(x, y, z) = (\beta \min(x, y)^{\rho} + (1 - \beta)z^{\rho})^{1/\rho}$$

Determine β from the benchmark, and find demands for x, y and z if the price of x doubles.

7. Consider a two-period model in which consumers maximizes the discounted present value of utility:

$$U(c_1, c_2) = \frac{c_1^{1-\theta}}{1-\theta} + \beta\frac{c_2^{1-\theta}}{1-\theta}$$

subject to the budget constraint:

$$c_1 + \frac{c_2}{1+r} = 1 + \frac{1}{1+r}$$

in which β is the discount factor, θ is the intertempoal elasticity parameter and r is the given interest rate.

Use the calibrated share formulation to show (on inspection) that the equivalent variation of a change in the interest rate from $\bar{r}$ to r is equal to:

$$EV = M/\bar{M} - 1 = \left(\frac{2+r}{2+\bar{r}}\right)\left(\frac{1+\bar{r}}{1+r}\right)\left(\frac{1+\beta^{1/\theta}(1+r)^{1/\theta-1}}{1+\beta^{1/\theta}(1+\bar{r})^{1/\theta-1}}\right)^{\theta/(1-\theta)} - 1$$

4 Flexibility and Non-Separable CES

We let π_i denote the user price of the ith input, and let $x_i(\pi)$ be the cost-minizing demand for the ith input. The reference price and quantities are $\bar{\pi}_i$ and $\bar{x}_i$. One can think of set i as $\{K, L, E, M\}$ but the methods we employ may be applied to any number of inputs. Define the reference cost, and reference value share for ith input by $\bar{C}$ and θ_i, where

$$\bar{C} \equiv \sum_i \bar{\pi}_i \bar{x}_i$$

and

$$\theta_i \equiv \frac{\pi_i \bar{x}_i}{\bar{C}}$$

The single-level constant elasticity of substitution cost function in calibrated form is written:

$$C(\pi) = \bar{C} \left(\sum_i \theta_i \left(\frac{\pi_i}{\bar{\pi}_i} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

Compensated demands may be obtained from Shephard's lemma:

$$x_i(\pi) = \frac{\partial C}{\partial \pi_i} \equiv C_i = \bar{x}_i \left(\frac{C(\pi)}{\bar{C}} \frac{\bar{\pi}_i}{\pi_i} \right)^{\sigma}$$

Cross-price Allen-Uzawa elasticities of substitution (AUES) are defined as:

$$\sigma_{ij} \equiv \frac{C_{ij} C}{C_i C_j}$$

where

$$C_{ij} \equiv \frac{\partial^2 C(\pi)}{\partial \pi_i \, \partial \pi_j} = \frac{\partial x_i}{\partial \pi_j} = \frac{\partial x_j}{\partial \pi_i}$$

For single-level CES functions:

$$\sigma_{ij} = \sigma \quad \forall i \neq j$$

The CES cost function exibits homogeneity of degree one, hence Euler's condition applies to the second derivatives of the cost function (the Slutsky matrix):

$$\sum_j C_{ij}(\pi) \; \pi_j = 0$$

or, equivalently:

$$\sum_j \sigma_{ij} \theta_j = 0$$

The Euler condition provides a simple formula for the diagonal AUES values:

$$\sigma_{ii} = \frac{-\sum_{j \neq i} \sigma_{ij} \theta_j}{\theta_i}$$

As an aside, note that convexity of the cost function implies that all minors of order 1 are negative, i.e. $\sigma_{ii} < 0 \quad \forall i$. Hence, there must be *at least one* positive off-diagonal element in each row of the AUES or Slutsky matrices. When there are only two factors, then the off-diagonals must be negative. When there are three factors, then only one pair of negative goods may be complements.

Let:

k index a second-level nest

s_{ik} denote the fraction of good i inputs assigned to the kth nest

ω_k denote the benchmark value share of total cost which enters through the kth nest

γ denote the top-level elasticity of substitution

σ^k denote the elasticity of substitution in the kth aggregate

$p_k(\pi)$ denote the price index associated with aggregate k, normalized to equal unity in the benchmark, i.e.:

$$p_k(\pi) = \left[\sum_i \frac{s_{ik}\theta_i}{\omega_k} \frac{\pi_i}{\bar{\pi}_i})^{1-\sigma^k} \right]^{\frac{1}{1-\sigma^k}}$$

The two-level nested, nonseparable constant-elasticity-of-substitution (NNCES) cost function is then defined as:

$$C(\pi) = \bar{C} \left(\sum_k \omega_k p_k(\pi)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

Demand indices for second-level aggregates are needed to express demand functions in a compact form. Let $z_k(\pi)$ denote the demand index for aggregate k, normalized to unity in the benchmark; i.e.

$$z_k(\pi) = \left(\frac{C(\pi)}{\bar{C}} \frac{1}{p_k(\pi)} \right)^{\gamma}$$

Compensated demand functions are obtained by differentiating $C(\pi)$. In this derivative, one term arise for each nest in which the commodity enters, so:

$$x_i(\pi) = \bar{x}_i \sum_k z_k(\pi) \left(\frac{p_k(\pi)\bar{\pi}_i}{\pi_i} \right)^{\sigma^k} = \bar{x}_i \sum_k \left(\frac{C(\pi)}{\bar{C}} \frac{1}{p_k(\pi)} \right)^{\gamma} \left(\frac{p_k(\pi)\bar{\pi}_i}{\pi_i} \right)^{\sigma^k}$$

Simple differentiation shows that benchmark cross-elasticities of substitution have the form:

$$\sigma_{ij} = \gamma + \sum_k \frac{(\sigma^k - \gamma) s_{ik} s_{jk}}{\omega_k}$$

Given the benchmark value shares θ_i and the benchmark cross-price elasticities of substitution, σ_{ij}, we can solve for values of s_{ik}, ω_k, σ^k and γ. A closed-form solution of the calibration problem is not always practical, so it is convenient to compute these parameters using a constrained nonlinear programming algorithm, CONOPT, which is available through GAMS, the same programming environment in which the equilibrium model is specified. Perroni and Rutherford [1995] prove that calibration of the NNCES form is possible for arbitrary dimensions whenever the given Slutsky matrix is negative semi-definite. The two-level ($N \times N$) function is flexible for three inputs; and although we have not proven that it is flexible for 4 inputs, the only difficulties we have encountered have resulted from indefinite calibration data points.

Two GAMS programs are listed below. The first illustrates two analytic calibrations of the three-factor cost function. The second illustrates the use of nonlinear programming to calibrate a four-factor cost function.(See Rutherford [1999] for an introduction to MPSGE.)

```
$TITLE  Two nonseparable CES calibrations for a 3-input cost function.

*       Model-specific data defined here:

SET     i  Production input aggregates / A,B,C /;  ALIAS (i,j);

PARAMETER

    theta(i)    Benchmark value shares /A 0.2, B 0.5, C 0.3/

    aues(i,j)   Benchmark cross-elasticities (off-diagonals) /
                        A.B     2
                        A.C   -0.05
                        B.C     0.5 /;

*       Use an analytic calibration of the three-factor CES cost
*       function:

ABORT$(CARD(i) <> 3) "Error: not a three-factor model!";

*       Fill in off-diagonals:

aues(i,j)$aues(j,i)  =  aues(j,i);

*       Verify that the cross elasticities are symmetric:

ABORT$SUM((i,j), ABS(aues(i,j)-aues(j,i))) " AUES values non-symmetric?";

*       Check that all value shares are positive:

ABORT$(SMIN(i, theta(i)) <= 0) " Zero value shares are not valid:",THETA;

*       Fill in the elasticity matrices:

aues(i,i) = 0; aues(i,i) = -SUM(j, aues(i,j)*theta(j))/theta(i); DISPLAY aues;

SET     n       Potential nesting /N1*N3/
        k(n)    Nesting aggregates used in the model
        i1(i)   Good fully assigned to first nest
        i2(i)   Good fully assigned to second nest
        i3(i)   Good split between nests;

SCALAR  assigned /0/;

PARAMETER
        esub(*,*)       Alternative calibrated elasticities
        shr(*,i,n)      Alternative calibrated shares
        sigma(n)        Second level elasticities
        s(i,n)          Nesting assignments (in model)
        gamma           Top level elasticity (in model);

*       First the Leontief structure:

esub("LTF","GAMMA") = SMAX((i,j), aues(i,j));
esub("LTF",n) = 0;
LOOP((i,j)$((aues(i,j) = esub("LTF","GAMMA"))*(NOT assigned)),
```

```
        i1(i) = YES;
        i2(j) = YES;
        assigned = 1;
);

i3(i) = YES$((NOT i1(i))*(NOT i2(i)));
DISPLAY i1,i2,i3;
LOOP((i1,i2,i3),
        shr("LTF",i1,"N1") = 1;
        shr("LTF",i2,"N2") = 1;
        shr("LTF",i3,"N1") = theta(i1)*(1-aues(i1,i3)/aues(i1,i2)) /
                     ( 1 - theta(i3) * (1-aues(i1,i3)/aues(i1,i2)) );
        shr("LTF",i3,"N2") = theta(i2)*(1-aues(i2,i3)/aues(i1,i2)) /
                     ( 1 - theta(i3) * (1-aues(i2,i3)/aues(i1,i2)) );
        shr("LTF",i3,"N3") = 1 - shr("LTF",i3,"N1") - shr("LTF",i3,"N2");
);
ABORT$(SMIN((i,n), shr("LTF",i,n)) < 0) "Benchmark AUES is indefinite.";

*       Now specify the two-level CES function:

esub("CES","GAMMA") = SMAX((i,j), aues(i,j));
ESUB("CES","N1") = 0;
LOOP((i1,i2,i3),
        shr("CES",i1,"N1") = 1;
        shr("CES",i2,"N2") = 1;
        esub("CES","N2") = (aues(i1,i2)*aues(i1,i3)-aues(i2,i3)*aues(i1,i1)) /
                           (aues(i1,i3)-aues(i1,i1));
        shr("CES",i3,"N1") =
                (aues(i1,i2)-aues(i1,i3)) / (aues(i1,i2)-aues(i1,i1));
        shr("CES",i3,"N2") = 1 - shr("CES",i3,"N1");
);
ABORT$(SMIN(n, esub("CES",n)) < 0)   "Benchmark AUES is indefinite?";
ABORT$(SMIN((i,n), shr("CES",i,n)) < 0) "Benchmark AUES is indefinite?";

PARAMETER       price(i)        Price indices used to verify calibration,
                aueschk(*,i,j)  Check of benchmark AUES values;

price(i) = 1;

$ontext

$MODEL:CHKCALIB

$SECTORS:
        Y       ! PRODUCTION FUNCTION
        D(i)

$COMMODITIES:
        PY      ! PRODUCTION FUNCTION OUTPUT
        P(i)    ! FACTORS OF PRODUCTION
        PFX     ! AGGREGATE PRICE LEVEL

$CONSUMERS:
        RA

$PROD:Y  s:gamma  k.tl:sigma(k)
```

```
        O:PY            Q:1
        I:P(i)#(k)      Q:(theta(i)*s(i,k))     k.TL:

$PROD:D(i)
        O:P(i)  Q:theta(i)
        I:PFX   Q:(theta(i)*price(i))

$DEMAND:RA
        D:PFX
        E:PFX   Q:2
        E:PY    Q:-1
$OFFTEXT
$SYSINCLUDE mpsgeset CHKCALIB

SCALAR delta /1.E-5/;
SET     function /ltf, ces/;
alias (i,ii);

LOOP(function,
        k(n) = YES$SUM(i, shr(function,i,n));
        gamma = esub(function,"GAMMA");
        sigma(k) = esub(function,k);
        s(i,k) = shr(function,i,k);
        loop(ii,
          price(j) = 1; price(ii) = 1 + delta;
$INCLUDE CHKCALIB.GEN
          SOLVE CHKCALIB USING MCP;
          aueschk(function,j,ii) = (D.L(j)-1) / (delta*theta(ii));
));
aueschk(function,i,j) = aueschk(function,i,j) - aues(i,j);
DISPLAY aueschk;

*       Evaluate the demand functions:

$LIBINCLUDE plot

SET pr Alternative price levels /pr0*pr10/;

PARAMETER
        demand(function,i,pr)   Demand functions
        dplot(pr,function)      Demand function comparison

loop(ii,
        LOOP(function,
          k(n) = YES$SUM(i, shr(function,i,n));
          gamma = esub(function,"GAMMA");
          sigma(k) = esub(function,k);
          s(i,k) = shr(function,i,k);
          LOOP(pr,
            price(j) = 1;
            price(ii) = 0.2 * ORD(pr);
$INCLUDE CHKCALIB.GEN
            SOLVE CHKCALIB USING MCP;
            demand(function,ii,pr) = D.L(ii);
            dplot(pr,function) = D.L(ii);
          );
```

```
        );

*       Display the comparisons:

$LIBINCLUDE PLOT dplot
);

DISPLAY demand;
```

```
$TITLE  Numerical calibration of Nested CES from KLEM elasticities

SET  i  Production input aggregates / K, L, E, M/;  ALIAS (i,j);

*       Model-specific data defined here:

PARAMETER
    theta(i)    Benchmark value shares /K 0.2, L 0.4, E 0.05, M 0.35/

    aues(i,j)   Benchmark cross-elasticities (off-diagonals) /
                        K.L     1
                        K.E     -0.1
                        K.M     0
                        L.E     0.3
                        L.M     0
                        E.M     0.1 /;

SCALAR epsilon          Minimum value share tolerance /0.001/;

*       Fill in off-diagonals:

aues(i,j)$aues(j,i)  =  aues(j,i);

*       Verify that the cross elasticities are symmetric:

ABORT$SUM((i,j), ABS(aues(i,j)-aues(j,i))) " AUES values non-symmetric?";

*       Check that all value shares are positive:

ABORT$(SMIN(i, theta(i)) le 0) " Zero value shares are not valid:",theta;

*       Fill in the elasticity matrices:

aues(i,i) = 0; aues(i,i) = -SUM(j, aues(i,j)*theta(j))/theta(i); DISPLAY aues;

*       Define variables and equations for NNCES calibration:

SET     n       Nests within the two-level NNCES function /N1*N4/,
        k(n)    Nests which are in use;

VARIABLES
        S(i,n)        Fraction of good I which enters through nest N,
        SHARE(n)        Value share of nest N,
        SIGMA(n)        Elasticity of substitution within nest N,
        GAMMA           Elasticity of substitution at the top level,
        OBJ             Objective function;

POSITIVE VARIABLES S, SHARE, SIGMA, GAMMA;

EQUATIONS
        SDEF(i)         Nest shares must sum to one,
        TDEF(n)         Nest share in total cost,
        ELAST(i,j)      Consistency with given AUES values,
        OBJDEF          Maximize concentration;

ELAST(i,j)$(ORD(i) > ORD(j))..
```

```
                aues(i,j) =E=   GAMMA +

                        SUM(k, (SIGMA(k)-GAMMA)*S(i,k)*S(j,k)/SHARE(k));

TDEF(k)..       SHARE(k) =E= SUM(i, theta(i) * S(i,k));

SDEF(i)..       SUM(n, S(i,n)) =E= 1;

*       Maximize concentration at the same time keeping the elasticities
*       to be reasonable:

OBJDEF..        OBJ =E= SUM((i,k),S(i,k)*S(i,k))

                        - SQR(GAMMA) - SUM(k, SQR(sigma(k)));

MODEL   CESCALIB /ELAST, TDEF, SDEF, OBJDEF/;

*       Apply some bounds to avoid divide by zero:

SHARE.LO(n) = epsilon;

SCALAR  solved  Flag for having solved the calibration problem /0/
        minshr  Minimum share in candidate calibration;

SET     tries   Counter on the number of attempted calibrations /T1*T10/;

OPTION SEED=0;

LOOP(tries$(NOT solved),

*       Initialize the set of active nests and the bounds:

        k(n) = YES;
        S.LO(i,n) = 0;          S.UP(i,n) = 1;
        SHARE.LO(n) = epsilon;  SHARE.UP(n) = 1;
        SIGMA.LO(n) = 0;        SIGMA.UP(n) = +INF;

*       Install a starting point:

        SHARE.L(k)  = MAX(UNIFORM(0,1), epsilon);
        S.L(i,k)    = UNIFORM(0,1);
        GAMMA.L     = UNIFORM(0,1);
        SIGMA.L(k)  = UNIFORM(0,1);
        SDEF.M(i)     = 0; TDEF.M(k)     = 0; ELAST.M(i,j)  = 0;

        SOLVE CESCALIB USING NLP MAXIMIZING OBJ;

        solved = 1$(CESCALIB.MODELSTAT LE 2);
        IF (solved,
          minshr = SMIN(k, SHARE.L(k)) - epsilon;
          IF (minshr EQ 0,
            k(n)$(SHARE.L(n) = epsilon) = NO;
            S.FX(i,n)$(not k(n)) = 0;
            SHARE.FX(n)$(not k(n)) = 0;
            SIGMA.FX(n)$(not k(n)) = 0;
```

```
            DISPLAY "Recalibrating with the following nests:",k;
            SOLVE CESCALIB USING NLP MAXIMIZING OBJ;
            IF (cescalib.modelstat gt 2, solved = 0;);
            minshr = SMIN(k, SHARE.L(k)) - epsilon;
            IF (minshr=0, solved = 0;);
          );
        );
);
IF (solved,  DISPLAY "Function calibrated:",GAMMA.L,SIGMA.L,SHARE.L,S.L;
ELSE         DISPLAY "Function calibration fails!";
);
```

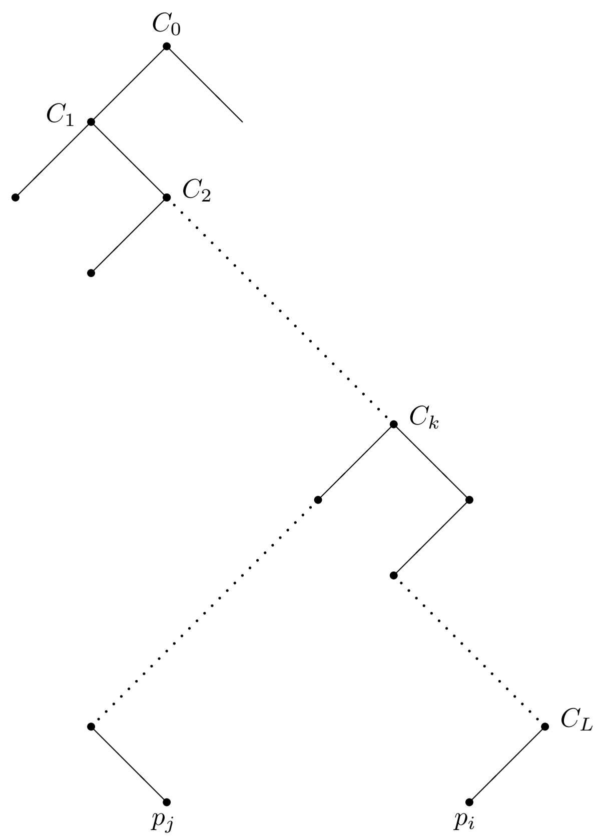

Figure 1: A Multi-level Nested CES Cost Function

5 Price Elasticities in Nested CES Functions

Suppose that we have a nested cost function of arbitrary depth and complexity. The prices for goods i and j, p_i and p_j are arguments of $C(p)$. Assume that the cost function $C(p)$ is *nested*. In the simplest two level case, we would have:

$$C(p) = \left(\sum_k \theta_k c_k(p)^{1-\sigma_0} \right)^{1/1-\sigma_0}$$

in which:

$$c_k(p) = \left(\sum_{i \in I_k} \alpha_{ik} p_i^{1-\sigma_k} \right)^{1/1-\sigma_k}$$

where I_k indicates the set of commodities entering nest k.

In a more general case, we could have cost aggregates as arguments to other cost aggregates. The figure shown below displays a graph of the nested CES cost function in which we number the nested cost functions which lead from the top level to nest containing good i as C_0 (top level), C_1, ..., C_L.

If we construct the cost function from a calibrated benchmark in which input prices and total cost are unity , we can scale the benchmark values of the subaggregate cost functions as unity and express the demand for good i as:

$$x_i = \bar{x}_i \left(\frac{C_L}{p_i}\right)^{\sigma_L} \left(\frac{C_{L-1}}{C_L}\right)^{\sigma_{L-1}} \cdots \left(\frac{C_0}{C_1}\right)^{\sigma_0} \;=\; \bar{x}_i \; p_i^{-\sigma_L} \; C_0^{\sigma_0} \prod_{n=1}^{L} C_n^{\sigma_n - \sigma_{n-1}}$$

5.1 The Own-Price Elasticity of Demand

By Shephard's lemma the derivative of C_n with respect to p_i equals the demand for good i per unit of aggregate n. Recalling that all prices are scaled to unity, the benchmark "quantity" of aggregate n equals the sum of the inputs which enter directly or indirectly into that cost function:

$$\bar{X}_n = \sum_{j \in I_n} \bar{x}_j$$

and

$$\left.\frac{\partial C_n}{\partial p_i}\right|_{p=1} = \begin{cases} 0 & i \notin I_n \\ \frac{\bar{x}_i}{\bar{X}_n} & i \in I_n \end{cases}$$

We then can compute the compensated own-price elasticity of demand for good i:

$$\eta_i \equiv \left.\frac{\partial x_i}{\partial p_i}\right|_{p=1} = -\sigma_L + \bar{x}_i \left(\sigma_0 + \sum_{n=1}^{L} \frac{\sigma_n - \sigma_{n-1}}{\bar{X}_n}\right)$$

5.2 The Cross-Price Elasticity of Demand

When we evaluate the elasticity of demand for i with respect to a change in the price of good j, we can let k denote the deepest price aggregate which contains both p_i and p_j. (See Figure 1).

The cross derivative can then be computed using the demand function for x_i, taking into account the impact of p_j on C_k, C_{k-1}, …, C_0:

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{p=1} = \bar{x}_i \left[\sigma_0 \frac{\partial C_0}{\partial p_j} + \sum_{n=1}^{k} (\sigma_n - \sigma_{n-1}) \frac{\partial C_n}{\partial p_j}\right]$$

One means of representing the dependence of x_i on p_j is with the Allen-Uzawa elasticity-of-substitution which:

$$\sigma_{ij} \equiv \frac{\partial x_i}{\partial p_j} \frac{C_0}{x_i x_j} = \sigma_0 + \sum_{n=1}^{k} \frac{\sigma_n - \sigma_{n-1}}{X_n}$$

As a logical check on this elasticity, consider two special cases:

1. $\sigma_n = \sigma_0 \quad \forall n$

   This then leads to single level CES, impling a constant cross elasticity of substitution between all input pairs.

2. $\sigma_n = 0 \quad \forall n < k$

   We then have Leontief demand for aggregate k, implying that the elasticity of substitution between i and j is given by

   $$\sigma_{ij} = \sigma_k$$

   N.B. The cross elasticity between i and j is independent of the subnest elasticity for all nests $n > k$.

6 Benchmarking Supply Functions

This section describes how calibrate the fixed factor input for a constant returns to scale CES technology and obtain an arbitrary price elasticity of supply at a reference point. For concreteness, consider output as a function of labor and capital inputs. Consider the labor input to be variable and the capital input to be fixed. We then have a CES cost function which in equilibrium defines the price of output:

$$p = c(r, w)$$

in which w is the exogenous wage rate and r is the residual return to the sector's fixed factor. Because this factor is fixed, by Shepard's lemma we have the following relationship between output, the supply of the fixed factor and the return to the fixed factor:

$$y \frac{\partial c(r, w)}{\partial r} = \bar{R}$$

If we use the calibrated CES cost function of the form:

$$c(r, w) = \left(\theta r^{1-\sigma} + (1-\theta) w^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

then the calibration problem consists of finding a values for θ and σ for which:

$$\frac{\partial y}{\partial (p/w)} \frac{(p/w)}{y} = \eta$$

at the benchmark point.

Note that we are free to choose units of the specific factor such that its benchmark price is unity. Hence, when we calibrate the share parameter, we are also determining the supply of the fixed factor:

$$\bar{R} = \theta \bar{y}$$

in which we scale the benchmark price of output to unity.

If the relative price of output and the variable factor depart from their benchmark values, the supply constraint for sector-specific can be inverted to obtain an explicit expression for the return:

$$r = p \left(\frac{\theta y}{\bar{R}}\right)^{1/\sigma}$$

where we have substituted the equilibrium price for the cost function. Substituting back into the cost function, we have

$$p^{1-\sigma} = \theta p^{1-\sigma} \left(\frac{\theta y}{\bar{R}}\right)^{\frac{1-\sigma}{\sigma}} + (1-\theta) w^{1-\sigma}$$

or

$$y = \bar{R} \theta^{\frac{1}{\sigma-1}} \left[1 - (1-\theta)\left(\frac{w}{p}\right)^{1-\sigma}\right]^{\frac{\sigma}{1-\sigma}}$$

Differentiating this expression with respect the relative price of output, and setting all prices equal to unit, we have:

$$\eta = \frac{\sigma(1-\theta)}{\theta}$$

This equation can be used in a variety of ways to calibrate the supply function. One approach would be to choose the value share of the fixed factor θ to match the base year profits, and then assign the elasticity according to:

$$\sigma = \frac{\theta \eta}{(1-\theta)}$$

Alternatively, one choose to use a Cobb-Douglas function and set the specific factor value share accordingly:

$$\theta = \frac{1}{1+\eta}$$

7 Calibration of Short- and Long-Run Elasticities

In a dynamic model it may be helpful to introduce two notions of the elasticity of supply: short-run and long-run. A simple way to introduce this distinction into a numerical model is to work with a three factor production function:

$$y = f(L, K, R)$$

where L is labor, a production factor which is variable in both the short and long run, K is capital, a quasi-fixed production factor which is variable in the long run but fixed in the short run, and R is a sector-specific resource which is fixed in both the short and long run.

We can write the long-run production function as:

$$y = \bar{y}\left[\theta_L\left(\frac{L}{\bar{L}}\right)^{\rho} + \theta_K\left(\frac{K}{\bar{K}}\right)^{\rho} + (1-\theta_L-\theta_K)\right]^{1/\rho}$$

The R input does not appear in the calibrated production function because we have assumed that $R = \bar{R}$. In the short-run model, we have:

$$y = \bar{y}\left[\theta_L\left(\frac{L}{\bar{L}}\right)^{\rho} + (1-\theta_L)\right]^{1/\rho}$$

If the short-run elasticity of supply is given by η_S, and the labor value share (θ_L) is given in the benchmark data, we can then calibrate the elasticity of substitution to match these inputs:

$$\sigma = \eta_S \frac{1-\theta_L}{\theta_L}$$

Let us assume that while the labor value share is observed (employment statistics are commonly available for many secttors and regions), but the allocation of the remaining value added is not known with certainty. We can take advantage of this uncertainty to calibrate the long-run supply response by choosing the capital value as:

$$\theta_K = \frac{\eta_L}{\sigma + \eta_L} - \theta_L$$

given the value of σ previously calibrated. (It can be shown that when $\eta_L > \eta_S$, we find that $\theta_K < 1 - \theta_L$.)

8 The GEMTAP Final Demand System

Following Ballard, Fullerton, Shoven and Whalley (BFSW), we consider a representative agent whose utility is based upon current consumption, future consumption and current leisure. Changes in *future consumption*; in this static framework are associated with changes in the level of savings. There are three prices which jointly determine the price index for future consumption. These are:

P_I the composite price index for investment goods

P_K the composite rental price for capital services

P_C the composite price of current consumption.

All of these prices equal unity in the benchmark equilibrium.

Capital income in each future year finances future consumption, which is expected to cost the same as in the current period, P_C (static expectations). The consumer demand for savings therefore depends not only on P_I, but also on P_K and P_C, namely:

$$P_S = \frac{P_I P_C}{P_K}$$

The price index for savings is unity in the benchmark period. In a counter-factual equilibrium, however, we would expect generally that

$$P_S \neq P_I$$

. When these price indices are not equal, there is a *virtual tax payment*; associated with savings demand.

Following BFSW, we adopt a nested CES function to represent preferences. In this function, at the top level demand for savings (future consumption) trades off with a second CES aggregate of leisure and current consumption. These preferences can be summarized with the following expenditure function:

$$P_U = \left[\alpha {P_H}^{1-\sigma_S} + (1-\alpha){P_S}^{1-\sigma_S}\right]^{\frac{1}{1-\sigma_S}}$$

Preferences are homothetic, so we have defined P_U as a linearly homogeneous cost index for a unit of utility. We conveniently scale this price index to equal unity in the benchmark. In this definition,α is the benchmark value share for current consumption (goods and leisure). P_H is a compositive price for current consumption defined as:

$$P_H = \left[\beta {P_\ell}^{1-\sigma_L} + (1-\beta){P_C}^{1-\sigma_L}\right]^{\frac{1}{1-\sigma_L}}$$

in which β is the benchmark value share for leisure within current consumption.

Demand functions are:

$$S = S_0 \left(\frac{P_U}{P_F}\right)^{\sigma_S} \frac{I}{I_0 P_U},$$

$$C = C_0 \left(\frac{P_H}{P_C}\right)^{\sigma_L} \left(\frac{P_U}{P_H}\right)^{\sigma_S} \frac{I}{I_0 P_U},$$

and

$$\ell = \ell_0 \left(\frac{P_H}{P_L}\right)^{\sigma_L} \left(\frac{P_U}{P_H}\right)^{\sigma_S} \frac{I}{I_0 P_U}.$$

Demands are written here in terms of their benchmark values (S_0, C_0 and ℓ_0) and current and benchmark income (I and I_0).

There are four components in income. The first is the value of labor endowment (E), defined inclusive of leisure. The second is the value of capital endowment (K). The third is all other income (M). The fourth is the value of *virtual tax revenue*; associated with differences between the shadow price of savings and the cost of investment.

$$I = P_L E + P_K K + M + (P_S - P_I)S$$

The following parameter values are specified exogenously:

$\zeta = 1.75$ is the ratio of labor endowment to labor supply, $\zeta \equiv \frac{E}{L_0}$, where L_0 is the benchmark labor supply. Labor supply and ζ also define benchmark leisure demand, $\ell_0 = L_0(\zeta - 1)$.

$\xi = 0.15$ is the uncompensated elasticity of labor supply with respect to the net of tax wage, i.e.

$$\xi = \frac{\partial L}{\partial P_L}\frac{P_L}{L} \;=\; \frac{\partial(E-\ell)}{\partial P_L}\frac{P_L}{L} \;=\; -\frac{\partial \ell}{\partial P_L}\frac{P_L}{L}$$

$\eta = 0.4$ is the elasticity of savings with respect to the return to capital:

$$\eta \equiv \frac{\partial S}{\partial P_K}\frac{S}{P_K}$$

Shephard's lemma applied at benchmark prices provides the following identities which are helpful in deriving expressions for η and ξ:

$$\frac{\partial P_U}{\partial P_H} = \alpha, \quad \frac{\partial P_U}{\partial P_S} = 1-\alpha, \quad \frac{\partial P_H}{\partial P_L} = \beta, \quad \frac{\partial P_H}{\partial P_C} = 1-\beta$$

It is then a relatively routine application of the chain rule to show that:

$$\xi = (\zeta - 1)\left[\sigma_L + \beta(\sigma_S - \sigma_L) - \alpha\beta(\sigma_S - 1) - \frac{E}{I_0}\right]$$

and

$$\eta = \sigma_S \alpha + \frac{K}{I_0}$$

The expression for η does not involve σ_L, so we may first solve for σ_S and use this value in determining σ_L:

$$\sigma_S = \frac{\eta - \frac{K}{I_0}}{\alpha}$$

and

$$\alpha_L = \frac{\frac{\xi}{\zeta-1} - \sigma_S\beta(1-\alpha) - \alpha\beta + \frac{E}{I_0}}{1-\beta}$$

```
$TITLE A Maquette Illustrating Labor Supply / Savings Demand Calibration

*       Exogenous elasticity:

SCALAR  XI      UNCOMPENSATED ELASTICITY OF LABOR SUPPLY /0.15/,
        ETA     ELASTICITY OF SAVINGS WRT RATE OF RETURN /0.40/,
        ZETA    RATIO OF LABOR ENDOWMENT TO LABOR SUPPLY /1.75/;

*       Benchmark data:

SCALAR  C0      CONSUMPTION /2.998845E+2/,
        S0      SAVINGS /70.02698974/,
        LS0     LABOR SUPPLY / 2.317271E+2/,
        K0      CAPITAL INCOME /93.46960577/,
        PL0     MARGINAL WAGE /0.60000000/;

*       Calibrated parameters:

SCALAR  EL0     LABOR ENDOWMENT
        L0      LEISURE DEMAND
        M0      NON-WAGE INCOME
        I       EXTENDED GROSS INCOME
        ETAMIN  SMALLEST PERMISSIBLE VALUE FOR ETA,
        XIMIN   SMALLEST PERMISSIBLE VALUE FOR XI,
        ALPHA   CURRENT CONSUMPTION VALUE SHARE
        BETA    LEISURE VALUE SHARE IN CURRENT CONSUMPTION
        SIGMA_L ELASTICITY OF SUBSTITUTION WITHIN CURRENT CONSUMPTION
        SIGMA_S ELASTICITY OF SUBSTITUTION - SAVINGS VS CURRENT CONSUMPTION
        TS      SAVINGS PRICE ADJUSTMENT;

LS0 = LS0 * PL0;
EL0 = ZETA * LS0;
L0 = EL0 - LS0;
M0 = C0 + S0 - LS0 - K0;
I = L0 + C0 + S0;
BETA =  L0 / (C0 + L0);
ALPHA = (L0 + C0) / I;
SIGMA_S = (ETA - K0 / I) / ALPHA;
ETAMIN  = K0 / I;
ABORT$(SIGMA_S LT 0) " Error: cannot calibrate SIGMA_S", ETAMIN;

SIGMA_L = (XI*(LS0/L0)-SIGMA_S*BETA*(1-ALPHA)-ALPHA*BETA+EL0/I)/(1-BETA);
XIMIN = -(L0/LS0) * (- SIGMA_S * BETA * (1-ALPHA) - ALPHA*BETA + EL0/I);
ABORT$(SIGMA_L LT 0) " Error: cannot calibrate SIGMA_L", XIMIN;

DISPLAY "Calibrated elasticities:", SIGMA_S, SIGMA_L;

$ONTEXT

$MODEL:CHKCAL

$COMMODITIES:
        PL
        PK
        PC
        PS
```

```
$SECTORS:
        Y
        S

$CONSUMERS:
        RA

$PROD:Y
        O:PC    Q:(K0+LS0-S0)
        I:PL    Q:(LS0-S0)
        I:PK    Q:K0

$PROD:S
        O:PS    A:RA  T:TS
        I:PL

$DEMAND:RA  s:SIGMA_S a:SIGMA_L
        E:PC    Q:M0
        E:PL    Q:EL0
        E:PK    Q:K0
        D:PS    Q:S0
        D:PC    Q:C0  a:
        D:PL    Q:L0  a:
$OFFTEXT
$SYSINCLUDE mpsgeset CHKCAL

S.L = S0;
TS  = 0;

*       VERIFY THE BENCHMARK:

CHKCAL.ITERLIM = 0;
$INCLUDE CHKCAL.GEN
SOLVE CHKCAL USING MCP;

PL.L = 1.001;

CHKCAL.ITERLIM = 0;
$INCLUDE CHKCAL.GEN
SOLVE CHKCAL USING MCP;

*       Compute induced changes in labor supply using the labor market
*       "marginal", PL.M.  This marginal returns the net excess supply of
*       labor at the given prices.  We started from a balanced benchmark,
*       with no change in labor demand (the iteration limit was zero).
*       Hence, PL.M returns the magnitude of the change in labor supply.
*       We multiply by the benchmark wage (1) and divide by the benchmark
*       labor supply (LS0) to produce a finite difference approximation
*       of the elasticity:

DISPLAY "CALIBRATION CHECK -- THE FOLLOWING VALUES SHOULD BE IDENTICAL:", XI;
XI = (PL.M / 0.001) * (1 / LS0);
DISPLAY XI;

PL.L = 1.0;
```

```
*       CHECK THE ELASTICITY OF SAVINGS WRT RENTAL RATE OF CAPITAL:

PK.L = 1.001;
PS.L = 1 / 1.001;
TS   = 1 / 1.001 - 1;

CHKCAL.ITERLIM = 0;

*       Compute elasticity of savings with respect to the rental rate of
*       capital.  This requires some recursion in order to account for the
*       effect of changes in savings on effective income.  When PK increases,
*       PS declines -- there is an effective "subsidy" for saving, paid from
*       consumer income.  In order to obtain a difference approximation for
*       the elasticity of savings response, we need to make sure the virtual
*       tax payments are properly handled.  In the MPSGE model, this means
*       that the level value for S must be adjusted so that it exactly equals
*       the savings.  We do this recursively:

SET ITER /IT1*IT5/;

PS.M = 1;
LOOP(ITER$(ABS(PS.M) GT 1.0E-8),

$INCLUDE CHKCAL.GEN
        SOLVE CHKCAL USING MCP;
        S.L = S.L - PS.M;
);

DISPLAY "CALIBRATION CHECK -- THE FOLLOWING VALUES SHOULD BE IDENTICAL:", ETA;
ETA = ((S.L - S0) / 0.001) * (1 / S0);
DISPLAY ETA;
```

9 Calibrating the Demand for Bequest

Bequests play an important part in the life-cycle income and expenditures of certain classes of households (see Kotlikoff and Summers [1981]). Bequests are important for analyzing the intergenerational impact of economic policy because they provide a mechanism through which impacts on older generations are transmitted to younger generations.

A simple means of introducing bequests to the OLG exchange model involves the introduction of a separate "bequest market" for each generation. A representative of generation g then demands two *goods*, $\hat{u}_g$ and b_g. The first of these is the composite of lifetime consumption as defined by the utility function:

$$\begin{aligned} \max_{c_{g,t}} u_g\left(c_{g,t}\right) &= \sum_{t=g}^{g+N} \left(\frac{1}{1+\rho}\right)^{t-g} \frac{c_{g,t}^{1-\theta}}{1-\theta} \\ s.t. \quad \sum_{t=g}^{g+N} p_t c_{g,t} &\leq \sum_{t=g}^{g+N} p_t \omega_{g,t}, \end{aligned}$$

The second represents the amount of bequests to future generations. In this approach, we can think about the value of bequests as a reduced form which is calibrated to two parameters: the average bequest rate, β, and the bequest elasticity, ξ. If wealth of generation g is $\bar{m}_g$, we have:

$$\beta = \frac{\bar{b}_g}{\bar{m}_g} \tag{1}$$

and

$$\xi = \left.\frac{\partial b_g}{\partial m_g}\frac{m_g}{b_g}\right|_{\bar{m}_g,\bar{b}_g}. \tag{2}$$

In order to make the reduced form representation, we model the bequests as a specialized transfer good that is demanded by generation g and caries the price p_g^b. Endowments of this commodity are fixed and sum to $\bar{b}_g$, hence p_g^b is an index of the aggregate transfer. The ownership of the bequest good then determines which generations receive the bequest and p_g^b, equal to unity in the baseline, determines the value of the bequest.

We write the utility function for a representative generation g as

$$u(b,\hat{u}) = \left[\beta^{\frac{1}{\nu}} b^{\frac{\nu-1}{\nu}} + (1-\beta)^{\frac{1}{\nu}} \hat{u}^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}} \tag{3}$$

so the elasticity of substitution, ν, can be used to calibrate the bequest elasticity. The "demand" for bequests arising from budget-constrained utility maximization on the baseline growth path provides the following equation:

$$\bar{b} = \frac{\beta m}{\beta p_b + (1-\beta)p_b^{\nu}}. \tag{4}$$

Evaluating $\partial m/\partial p_b$ permits us to express ξ as a function of ν, and we may the invert this relationship to find the elasticity of substitution as a function of the average bequest rate and the bequest elasticity:[2]

$$\nu = \frac{1-\xi\beta}{\xi(1-\beta)}. \tag{5}$$

[2]When the bequest elasticity is unity the ratio of bequest to wealth is constant, and (3) is replaced by a Cobb-Douglas function with β as the value share of bequests.

10 Alternative Functional Forms

A well known dimensionless index of second-order curvature is the compensated price elasticity (CPE), which is defined as:[3]

$$\sigma_{ij}^{C} \equiv \frac{\partial \ln C_i}{\partial \ln p_j} = \frac{C_{ij} p_j}{C_i},$$

and a related measure of second-order curvature is the AUES, which has been already discussed. This can also be written as

$$\sigma_{ij}^{A} = \frac{\sigma_{ij}^{C}}{\theta_j}.$$

The AUES is a *one-input-one-price* elasticity of substitution (Mundlak, 1968), since, as the definition of σ_{ij}^{A} makes clear, it measures the responsiveness of the compensated demand for one input to a change in one input price. In contrast, the Morishima elasticity of substitution (MES; Morishima, 1967) constitutes a *two-input- one-price* elasticity measure, being defined as

$$\sigma_{ij}^{M} \equiv \frac{\partial \ln(C_i/C_j)}{\partial \ln(p_i/p_j)} = \sigma_{ij}^{C} - \sigma_{jj}^{C}.$$

Note that, in general, the MES is not symmetric, i.e. $\sigma_{ij}^{M} \neq \sigma_{ji}^{M}$.

A third type of curvature measure is represented by the class of *two-input-two-price* elasticities of substitution, which take the form $\partial \ln(C_i/C_j)/\partial \ln(p_j/p_i)$. One such index is the shadow elasticity of substitution (SES; Frenger, 1985), which is defined as

$$\sigma_{ij}^{S} \equiv \frac{\theta_i \sigma_{ij}^{M} + \theta_j \sigma_{ji}^{M}}{(\theta_i + \theta_j)}.$$

When technologies are of the CES type, σ_{ij}^{A}, σ_{ij}^{M} and σ_{ij}^{S} are all identical, but they are generally different otherwise.

10.1 The Translog Cost Function

The Translog unit cost function is defined as

$$\ln C(p) \equiv \ln\, b_0 + \sum_i b_i \ln\, p_i + \frac{1}{2} \sum_{ij} a_{ij} \ln\, p_i\, \ln\, p_j \equiv \ln\, b_0 + L(p).$$

Compensated Demand Functions:

$$x_i(p) = \frac{\partial C(p)}{\partial p_i} = \frac{\partial \ln\, C(p)}{\partial p_i} C = \frac{C(p)}{p_i} \left[b_i + \sum_j a_{ij} \ln\, p_j \right]$$

Restrictions:

$$\sum_i b_i = 1;$$

$$a_{ij} = a_{ji}, \qquad \forall i, \forall j;$$

$$\sum_j a_{ij} = 0, \qquad \forall i.$$

[3]This exposition is based on Perroni Rutherford, 1998.

Calibration:

$$a_{ij} = \theta_i \theta_j (\sigma^A_{ij} - 1), \qquad i \neq j;$$
$$a_{ii} = -\sum_{j \neq i} a_{ij}, \qquad \forall i;$$
$$b_i = \theta_i - \sum_j a_{ij} \ln p_j, \qquad \forall i;$$
$$b_0 = \bar{C} e^{-L(p)}.$$

10.2 The Generalized Leontief Cost Function

The Generalized Leontief unit cost function is defined as

$$C(p) \equiv \frac{1}{2} \sum_{ij} a_{ij} \sqrt{p_i p_j}.$$

Compensated Demand Functions:

$$x_i(p) = \frac{\partial C(p)}{\partial p_i} = \sum_j \frac{a_{ij}}{2} \sqrt{\frac{p_j}{p_i}}.$$

Restrictions:

$$a_{ij} = a_{ji}, \qquad \forall i, \forall j.$$

Calibration:

$$a_{ij} = 4\sigma^A_{ij} \ \theta_i \theta_j \frac{\bar{C}}{\sqrt{p_i p_j}} \ , \qquad i \neq j;$$
$$a_{ii} = 2\theta_i \frac{\bar{C}}{p_i} - \sum_{j \neq i} a_{ij} \sqrt{\frac{p_j}{p_i}} \qquad \forall i.$$

10.3 The Normalized Quadratic Cost Function

The Normalized Quadratic unit cost function is defined as

$$C(p) \equiv \frac{1}{2} \frac{\sum_{ij} a_{ij} p_i p_j}{\sum_i b_i p_i}.$$

Compensated Demand Functions:

$$x_i(p) = \frac{\partial C(p)}{\partial p_i} = \frac{\sum_j a_{ij} p_j - C(p) b_i}{\sum_j b_j p_j}$$

Restrictions:

$$a_{ij} = a_{ji}, \qquad \forall i, \forall j;$$
$$b_i \geq 0, \qquad \forall i;$$
$$\sum_i b_i = 1.$$

Calibration:

$$a_{ij} = \frac{\bar{C}\theta_i\theta_j}{p_i p_j} \left(\sigma_{ij}^A \sum_k b_k p_k + \frac{b_i p_i}{\theta_i} + \frac{b_j p_j}{\theta_j} \right), \qquad \forall i, \forall j;$$

We examined two alternative specifications, one in which $b_i = \theta_i$, and another in which $b_i = 1/N$. The first specification is reported by Perroni and Rutherford [1998] to produce a more stable function.

10.4 The Nonseparable Nested CES Cost Function

We restrict our discussion to the case $N = 3$ (for the general N-input case see Perroni and Rutherford, 1995), and focus on a particular nesting structure which we call "Lower Triangular Leontief" (LTL). Let us rearrange indices so that the maximum off-diagonal AUES element is σ_{12}^A. Then the three-input NNCES-LTL cost function can be defined as

$$C(p) \equiv \phi \left[\alpha (a_1 p_1 + a_3 p_3)^{1-\gamma} + (1-\alpha)(b_2 p_2^{1-\mu} + b_3 p_3^{1-\mu})^{\frac{1-\gamma}{1-\mu}} \right]^{\frac{1}{1-\gamma}}.$$

Compensated Demand Functions: We simplify the algebra by defining price indices for the two nests:

$$p_{13} = a_1 p_1 + a_3 p_3$$

and

$$p_{23} = \left[b_2 p_2^{1-\mu} + b_3 p_3^{1-\mu} \right]^{\frac{1}{1-\mu}}$$

we have:

$$x_1(p) = \frac{\partial C(p)}{\partial p_1} = a_1 \alpha \phi \left(\frac{C(p)}{\phi p_{13}} \right)^{\gamma}$$

$$x_2(p) = \frac{\partial C(p)}{\partial p_2} = b_2 (1-\alpha) \phi \left(\frac{C(p)}{\phi p_{23}} \right)^{\gamma} \left(\frac{\phi p_{23}}{p_2} \right)^{\mu}$$

$$x_3(p) = \frac{\partial C(p)}{\partial p_3} = a_3 \alpha \phi \left(\frac{C(p)}{\phi p_{13}} \right)^{\gamma} + b_3 (1-\alpha) \phi \left(\frac{C(p)}{\phi p_{23}} \right)^{\gamma} \left(\frac{\phi p_{23}}{p_3} \right)^{\mu}$$

Restrictions:

$$\gamma \geq 0;$$
$$\mu \geq 0;$$
$$\phi \geq 0;$$
$$a_i \geq 0, \qquad \forall i;$$
$$b_i \geq 0, \qquad \forall i.$$

Calibration:

Let us denote with s_3 the fraction of the total input of commodity 3 which enters the first subnest of the structure:

$$C(p) \equiv \phi \left[\alpha (a_1 p_1 + a_3 p_3)^{1-\gamma} + (1-\alpha)(b_2 p_2^{1-\mu} + b_3 p_3^{1-\mu})^{\frac{1-\gamma}{1-\mu}} \right]^{\frac{1}{1-\gamma}}.$$

(with $(1 - s_3)$ representing the fraction entering the second subnest). If we select

$$\gamma = \sigma^A_{12};$$

$$\mu = \frac{\sigma^A_{12}\sigma^A_{13} - \sigma^A_{23}\sigma^A_{11}}{\sigma^A_{13} - \sigma^A_{11}};$$

it can be shown that

$$s_3 = \frac{\sigma^A_{12} - \sigma^A_{13}}{\sigma^A_{12} - \sigma^A_{11}}.$$

The remaining parameters can then be recovered as follows:

$$\phi = \bar{C};$$

$$\alpha = \theta_1 + s_3\theta_3$$

$$a_1 = \frac{\theta_1}{\alpha p_1}$$

$$a_3 = \frac{s_3\theta_3}{\alpha p_3}$$

$$b_2 = \frac{\theta_2}{(1-\alpha)p_2^{1-\mu}}$$

$$b_3 = \frac{(1-s_3)\theta_3}{(1-\alpha)p_3^{1-\mu}}$$

References

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